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On the stability of plane flow of a conducting fluid in the presence of a coplanar magnetic field

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The equations governing the propagation of small perturbations to plane flow of a viscous incompressible conducting fluid are re-examined with special reference to the case when the constant unperturbed magnetic field and flow velocity are parallel. We use the relationship between two parameters in one equation and, without computations, show the following: If for a non-zero value of the Alfvén number the flow is unstable when the Reynolds and magnetic Reynolds numbers take particular finite values, then, for that value of the Alfvén number, the flow cannot be completely stabilized for all finite Reynolds numbers, when the magnetic Reynolds number is finite. Since for a finite Alfvén number one expects that unstable flow cannot be stabilized for all finite Reynolds numbers, unless the magnetic Reynolds number exceeds some value, we deduce the following: An unstable parallel flow of a finitely conducting fluid cannot be completely stabilized for all finite Reynolds numbers by a constant magnetic field, which is coplanar with the flow.

1. Introduction

It is well known (Chandrasekhar 1961) that a suitable magnetic field tends to suppress the instability of various flows and in many cases it makes an otherwise unstable flow completely stable to small perturbations. For this reason many authors have investigated the stability effect of a magnetic field on various unstable flows. Plane Poiseuille flow of a viscous fluid between two fixed planes is known to be unstable to infinitesimal perturbations. Squire (1933) showed that the most unstable infinitesimal perturbations of this flow are two-dimensional.

The stability of conducting plane Poiseuille flow in the presence of a magnetic field was investigated by several authors. Michael (1953) investigated purely magnetic perturbations and showed that a steady flow and a constant magnetic field parallel to the flow are not unstable to two-dimensional disturbances. Stuart (1954), Velikhov (1959) and Tarasov (1960) investigated the stability of the problem when both the velocity and the magnetic field, which is assumed to be constant and parallel to the flow, are perturbed. They showed that a sufficiently strong magnetic field stabilizes two-dimensional disturbances and, as it was recently pointed out by Hunt (1966), they incorrectly concluded that Squire's theorem applied and deduced that such a magnetic field stabilizes the flow to all infinitesimal disturbances. The case when a constant magnetic field is parallel to the fixed planes and inclined at an angle to the flow was investigated by Wooler (1961). Wooler showed that for this configuration there are three-dimensional disturbances which are unstable. However, he also deduced that a sufficiently strong magnetic field aligned with the flow stabilizes all infinitesimal perturbations.

It was Hunt (1966) who showed that if the magnetic Reynolds number R_m is small, a constant magnetic field aligned with the flow cannot completely stabilize it. Using Tarasov's (1960) results Hunt indicated that a constant magnetic field parallel to the flow cannot stabilize it completely if R_m is finite.

Here we re-examine the case when the constant magnetic field is coplanar with the flow. We deduce that when R_m is finite the effect of the magnetic field is negligible on waves propagating in a direction close to the normal to the magnetic field. Thus these waves cannot be stabilized by the presence of a magnetic field. We also show that if R_m is infinite then a magnetic field, which is parallel to the flow and stabilizes two-dimensional disturbances, stabilizes three-dimensional disturbances as well.

Our analysis below applies to plane parallel flows more general than Poiseuille, such as flow between two planes in relative parallel motion and flows of the boundary-layer type.

2. Equations of the problem

We choose a Cartesian co-ordinate system (x_1, x_2, x_3) and consider the motion of a viscous electrically conducting incompressible fluid. Let ρ be the density, σ the electrical conductivity, ν the kinematic viscosity and $\mathbf{V}$ the velocity of the fluid. A solution of the equations of motion, in the steady state, satisfying appropriate boundary conditions is

$$\mathbf{V} = [U_0(x_2), 0, 0], \quad \mathbf{B} = \mathbf{B}_0 = (B_1, 0, B_3) \tag{1}$$

where $\mathbf{B}$ is the magnetic field and B_1 and B_3 are constants.

If we consider an infinitesimal perturbation of the steady state of the form

$$\left.\begin{aligned} \mathbf{v} &= [u(x_2), v(x_2), w(x_2)] \\ \mathbf{b} &= [\phi(x_2), \psi(x_2), \chi(x_2)] \end{aligned}\right\} \exp[i(\alpha x_1 + \gamma x_3 - \omega t)], \tag{2}$$

where $\mathbf{v}$ is the perturbation velocity and $\mathbf{b}$ the perturbation magnetic field, we find that v and ψ satisfy the following pair of equations (Stuart 1954)

$$(U-c)(v''-\lambda_1^2 v) - vU'' - \frac{b_0\lambda_1}{4\pi\rho\alpha_1 V_0}(\psi''-\lambda_1^2\psi) = -\frac{i}{\alpha_1 R}(v^{\text{iv}} - 2\lambda_1^2 v'' + \lambda_1^4 v) \tag{3}$$

and

$$(U-c)\psi - \frac{\lambda_1 b_0}{\alpha_1 V_0} v = -\frac{i}{\alpha_1 R_m}(\psi''-\lambda_1^2\psi). \tag{4}$$

In (3) and (4) we have set

$$x_2 = \delta y, \quad \omega/\alpha V_0 = c, \quad \alpha\delta = \alpha_1, \quad \gamma\delta = \gamma_1, \quad \lambda_1^2 = \alpha_1^2 + \gamma_1^2,$$

$$U_0 = UV_0, \quad R = V_0\delta/\nu, \quad R_m = 4\pi\sigma V_0\delta, \quad b_0 = (\alpha_1 B_1 + \gamma_1 B_3)/\lambda_1$$

and primes denote differentiation with respect to y. V_0 is a characteristic velocity of the flow and δ a characteristic length.

If we let $\theta[=\tan^{-1}(\gamma_1/\alpha_1)]$ denote the angle between the velocity vector and the direction of propagation of the wave and $\phi[=\tan^{-1}(B_3/B_1)]$ denote the angle between the magnetic field and the velocity vector, (3) and (4) become (Wooler 1961)

$$(U-c)(v''-\lambda_1^2 v)-vU''-B_1\frac{(1+\tan\theta\tan\phi)}{4\pi\rho V_0}(\psi''-\lambda_1^2\psi)$$
$$=-\frac{i}{\lambda_1\bar{R}}(v^{\text{iv}}-2\lambda_1^2 v''+\lambda_1^4 v), \quad (5)$$

$$(U-c)\psi-B_1\frac{(1+\tan\theta\tan\phi)}{V_0}v=-\frac{i}{\lambda_1\bar{R}_m}(\psi''-\lambda_1^2\psi), \quad (6)$$

where $\alpha_1 R=\lambda_1\bar{R}$ and $\alpha_1 R_m=\lambda_1\bar{R}_m$ or $\bar{R}=R\cos\theta$ and $\bar{R}_m=R_m\cos\theta$. When γ_1 and θ are equal to zero, (5) and (6) reduce to

$$(U-c)(v''-\alpha_1^2 v)-vU''-\frac{B_1}{4\pi\rho V_0}(\psi''-\alpha_1^2\psi)=-\frac{i}{\alpha_1 R}(v^{\text{iv}}-2\alpha_1^2 v''+\alpha_1^4 v) \quad (7)$$

and

$$(U-c)\psi-\frac{B_1 v}{V_0}=-\frac{i}{\alpha_1 R_m}(\psi''-\alpha_1\psi). \quad (8)$$

If we eliminate v between (5) and (6) and set $\phi=0$ we obtain,

$$\left[(U-c)^2-A^2-\frac{iU''}{\lambda_1\bar{R}_m}\right](\psi''-\lambda_1^2\psi)+2U'(U-c)\psi'$$
$$=-i\left[\frac{(D^2-\lambda_1^2)^2(U-c)\psi}{\lambda_1\bar{R}}+\frac{(U-c)(D^2-\lambda_1^2)^2\psi}{\lambda_1\bar{R}_m}\right]+\frac{(D^2-\lambda_1^2)^3\psi}{\lambda_1^2\bar{R}\bar{R}_m}, \quad (9)$$

where $A[=B_1/(4\pi\rho)^{\frac{1}{2}}V_0]$ is the Alfvén number and $D=d/dy$.

3. Discussion

Wooler (1961) pointed out that equations (5) and (6) are very similar to equations (7) and (8) that govern the motion of waves propagating parallel to the direction of flow. λ_1, $\bar{R}$, $\bar{R}_m$ and B_1 $(1+\tan\theta\tan\phi)$ play the same role in (5) and (6) as α_1, R, and B_1 in (7) and (8). $B_1\cos\theta(1+\tan\theta\tan\phi)$ and B_1 are the components of the magnetic field along the direction of propagation for the two cases.

The component of the magnetic field perpendicular to the direction of propagation does not enter into the calculations and thus it does not affect the stability of the problem, no matter what the electrical conductivity of the fluid is. Thus it follows that if the magnetic field is almost perpendicular to the direction of wave propagation its effect on the stability of the problem is minimal. This is an important point and when R_m is finite, it does not depend on ϕ. If R_m is infinite this is true only when ϕ is different from zero.

It was pointed out by Wooler (1961) that for a given ϕ, say ϕ_1, we can find a value of θ, θ_1, so that $1+\tan\phi_1\tan\theta_1$ vanishes. Equations (5) and (6) are then decoupled and the magnetic field does not affect propagation in the direction θ_1. We can then find an R, say R_1 so that $\bar{R}_1 = R_{\text{crit}}$, where R_{crit} is the minimum value of R for which the flow, in the absence of the magnetic field, becomes unstable. The flow is then unstable to waves propagating in the direction of θ_1 for all $R > R_1$. We can obviously choose a direction of propagation θ, slightly different from θ_1, so that the coefficient of

$$\psi'' - \lambda_1^2 \psi \tag{10}$$

in (5) is very small and the effect of the magnetic field terms in (5) is minimal, in agreement with what was stated above. This argument apparently breaks down when $\phi_1 = 0$ or π because θ_1 has to be $\pm\frac{1}{2}\pi$ and there are not any unstable disturbances propagating at right angles to the direction of flow.

If $\phi = 0$ and $\theta \neq \pm\frac{1}{2}\pi$ the coefficient of the expression (10) in (5) is constant. However, for finite values of R_m when θ is sufficiently close to the normal to the magnetic field $\bar{R}_m$ is small and (6) shows that the expression (10) itself must be small. Thus we can make $\bar{R}_m$, and the effect of the magnetic field on the waves propagating in the direction θ, very small by choosing a θ sufficiently close to $\frac{1}{2}\pi$. Thus these waves will not be affected by the magnetic field and for some finite large R which depends on R_m will be unstable, if the flow, in the absence of a magnetic field, is unstable. Our argument can be seen in terms of equation (9) as follows:

If for a certain A, say A_0, there is an R_m, say $R_{m0}(A_0) \neq 0$, such that the disturbances described by (9) are unstable when $\theta = 0$ and R is greater than some finite value R^0_{crit}, it follows that for that A the disturbances described by (9) are unstable for all $\bar{R} > R^0_{\text{crit}}$, when $\bar{R}_m = R_{m0}$. Since

$$\bar{R} = R\cos\theta \quad \text{and} \quad R_m = R_m \cos\theta,$$

it follows that waves propagating along the direction θ, where

$$\theta = \cos^{-1}(R_{m0}/R_m); \quad R_m > R_{m0}, \tag{11}$$

are unstable when

$$R > R_{\text{crit}} = \frac{R_m R^0_{\text{crit}}}{R_{m0}}; \quad A \leqslant A_0; \quad R_m \geqslant R_{m0}. \tag{12}$$

Since for a given R_m and a given flow one expects that a larger A would be a more stabilizing factor, the first inequality (12) must hold for all $A \leqslant A_0$. Similarly, for a given flow and a given A, one expects that a larger R_m will be a more stabilizing factor and therefore the flow will be unstable when

$$R > R_{\text{crit}} = R^0_{\text{crit}}; \quad A \leqslant A_0; \quad R_m \leqslant R_{m0}; \quad \theta = 0.$$

As an example consider the case of plane Poiseuille flow (Stuart 1954). For small R_m, $R^0_{\text{crit}} = 5{\cdot}5 \times 10^3$ when $R_m A^2 = 0{\cdot}01$. If we take $A^2 = 10$, R_{m0} is $0{\cdot}001$. Therefore waves propagating in the direction $\cos^{-1}(0{\cdot}001/R_m)$ are unstable when

$$R > R_{\text{crit}} = 5{\cdot}5 \times 10^6 R_m; \quad A^2 \leqslant 10; \quad R_m \geqslant 0{\cdot}001.$$

Note that when R_m is of order unity or larger the direction of propagation θ, for these unstable waves, is close to the normal to the magnetic field.

Hunt (1966) showed graphically how, by using two-dimensional solutions [those for (7) and (8)], one could find the minimum critical R and the corresponding value of θ for a given R_m and A, and deduced that a parallel magnetic field cannot stabilize the flow completely. However, he restricted his analysis to the case when R_m is small and the first term on the left-hand side of (6) is negligible. He also showed that, by using two-dimensional solutions for a given A, we could find the minimum critical Reynolds number for an arbitrary R_m. Using the results of Tarasov (1960), who investigated two-dimensional disturbance for arbitrary A when R_m is of order unity, Hunt indicated also that a parallel magnetic field cannot stabilize a plane Poiseuille flow when R_m is finite.

Hunt's argument in the case of arbitrary R_m is a little involved and the difficulty with his analysis is that, for a given flow and A, one cannot decide whether the flow is stable or unstable for finite R_m, unless one knows the functional dependence of R_{crit} on R_m (the shape of the $R_{\text{crit}} - R_m$ curve as Hunt calls it) for two-dimensional disturbances. Using Tarasov's results Hunt concludes that, for a specified A, if R_m is greater than some value then R_{crit} is proportional to R_m. The constant of proportionality, not given by Hunt but easily deduced from the way the straight line $R_{\text{crit}} - R_m$ is set up, is the minimum of R^0_{crit}/R_{m0}.

With our argument, however, for a given flow and a given A we need only find an arbitrary pair R_{m0}, R^0_{crit} (= finite) and then use (11) in order to specify a direction along which the waves cannot propagate stably for all finite R when R_m takes arbitrary finite values. Thus, it can be deduced straight away from Stuart's results, and without the need to look into Tarasov's data, that plane Poiseuille flow cannot be stabilized for all finite R when A and R_m are finite. To show, for example, that the case $R_m = 10$, $A_0^2 = 100$ is unstable, we take $R_{m0} = 0{\cdot}0001$. Stuart's table 2 shows that when $q = R_{m0}A_0^2 = 0{\cdot}01$, $R^0_{\text{crit}} = 5{\cdot}5 \times 10^3$. It then follows from (11) and (12) that when R_m is 10 the flow is unstable when R exceeds $5{\cdot}5 \times 10^8$. Since (12) holds for all $A \leqslant A_0$, we are in fact showing, by using Stuart's results, that the flow is not completely stable for small or large A and large R_m (= 10 in the present example), a case for which Stuart's analysis is not applicable. Note also that for a fixed A, Stuart's data (his table 2) show that the maximum value of R^0_{crit}/R_{m0} is only about three times as large as its minimum value. Thus, when R_m is small and Stuart's analysis applies, if we randomly use Stuart's table 2, our R_{crit} given by (12) will differ from the minimum R_{crit} by not more than a factor of three, when $R_m A^2$ is large ($> 0{\cdot}05$) and R_{crit}, according to Hunt's graphical method, becomes proportional to $R_m A^2$.

Indeed, our argument implies that if we assume that for a specified flow and A there exists a pair of values R_{m0} ($\neq 0$), R^0_{crit} (= finite), we are in fact assuming, that for this configuration, there can be found a direction $\theta(R_m)$ along which disturbances cannot propagate stably for all finite R when R_m ($\geqslant R_{m0}$) is finite. It seems plausible to us to assume that for a given arbitrary finite A an unstable flow cannot be stabilized for all finite R, unless R_m exceeds some non-zero value, say $R_{m0}(A)$. We, therefore, deduce that an unstable parallel flow of a finitely conducting fluid cannot be stabilized completely for all finite Reynolds numbers by the pressure of an aligned magnetic field. Hunt (1966) reached the same conclusion. His conclusion, however, was based on the assumption that the shape of the $R^0_{crit} - R_m$ curve, for constant A and all velocity profiles at all A, is similar to the one obtained by Stuart for small R_m for the case of plane Poiseuille flow. This is a much more stringent condition than the one we use.

Note that R_{m0} depends on A_0. As A_0 increases one expects R_{m0} to decrease and the minimum of R^0_{crit}/R_{m0}, which is used in Hunt's analysis when R_m exceeds a certain unspecified value, to increase. Therefore for a fixed R_m as A_0 increases, R_{crit} and θ increase. In the case of Stuart's work Hunt's graphical method gives

$$\theta = \cos^{-1}\left(\frac{0{\cdot}05}{R_m A^2}\right), \quad \text{for} \quad R_m A^2 \geqslant 0{\cdot}05,$$

and thus when $R_m A^2$ is large θ is close to the normal to the magnetic field.

Similarly for a fixed A as R_m increases so does R_{crit}. As R_m tends to infinity R_{crit} also tends to infinity and our argument breaks down. However, when R_m is infinite equation (6) becomes

$$(U-c)\,\psi = (B_1/V_0)\,(1+\tan\theta\tan\phi)\,v. \tag{13}$$

If $\phi = 0$, (13) shows that the v and ψ do not depend explicitly on θ, except in the trivial case $\theta = \frac{1}{2}\pi$, when $\psi = v = 0$. Thus if we eliminate ψ between (5) and (13) and set ϕ equal to zero, we obtain an equation for which Squire's theorem, that two-dimensional disturbances are the most unstable, is applicable. Thus when R_m is infinite and B_3 is zero, the value of A that stabilizes two-dimensional disturbances stabilizes the flow completely.

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